MIXED CHARACTERISTIC REDUCTION AND BCM-REGULARITY

TATSUKI YAMAGUCHI

BCM-regularity, introduced by Ma-Schwede [MS21], is a mixed characteristic analogue of klt and strong F-regular singularities. They are known to exhibit properties similar to F-regular singularities, but no effective criterion such as Fedder's criterion is known. In this talk, we introduce mixed characteristic reduction, a mixed charcteristic analogue of reduction modulo $p \gg 0$ and show BCM-regularity of mixed characteristic reductions for $p \gg 0$ in some case.

Definition 1. Let $(R, \mathfrak{m})$ be an excellent $\mathbb{Q}$-Gorenstein normal local domain.

(1) R is said to be *BCM-regular* if $R \to B$ is pure for any big Cohen-Macaulay R^+-algebra B.
(2) If $R/\mathfrak{m}$ is of characteristic $p > 0$, then R is said to be *perfectoid BCM-regular* if $R \to B$ is pure for any integral perfectoid big Cohen-Macaulay R^+-algebra B.

They satisfy following properties.

Proposition 2 ([MS21])**.** *Let R be an excellent $\mathbb{Q}$-Gorenstein normal local domain.*

(1) If R is BCM-regular, then R has log terminal singularities.
(2) Suppose that R is of characteristic $p > 0$. R is F-regular if and only if R is BCM-regular.

The following is expected to hold.

Conjecture 3 (cf. [MS21, Conjecture 3.9])**.** *Suppose that R is essentially of finite type over $\mathbb{C}$. R is log terminal if and only if R is BCM-regular.*

We clarify our setting.

Setting 4. Let $R := (\mathbb{C}[t, x_2, \ldots, x_n]/(f_1, \ldots, f_m))_{(t,x_2,\ldots,x_n)}$, where $f_1, \ldots, f_m \in (t, x_2, \ldots, x_n)\mathbb{Z}[t, x_2, \ldots, x_n]$, and let $R_p := (\mathbb{Z}_p[t, x_2, \ldots, x_n]/(t-p, f_1, \ldots, f_m))_{(p,x_2,\ldots,x_n)}$. We say that R_p is a mixed characteristic reduction of R.

Example 5. Let $R := (\mathbb{C}[t, x, y]/(t^2 + x^3 + y^5))_{(t,x,y)}$. Then

$$R_p := (\mathbb{Z}_p[x, y]/(p^2 + x^3 + y^5))_{(p,x,y)}$$

is a mixed characteristic reduction.

Theorem 6. *With notation as in Setting 4, suppose that R is BCM-regular. Then $\widehat{R_p}$ is perfectoid BCM-regular for $p \gg 0$.*

We can also show the following version.

Theorem 7. *With notation as in Setting 4, let S be a regular local ring essentially of finite type over $\mathbb{C}[t]_{(t)}$ such that t-torsion free and the residue field of S equals $\mathbb{C}$. Suppose that there exists a pure local $\mathbb{C}[t]_{(t)}$-algebra homomorphism $R \to S$. Then $\widehat{R_p}$ is perfectoid BCM-regular for almost all p.*

Example 8. Let $R := (\mathbb{C}[t, x, y]/(t^2 + x^2 + y^2))_{(t,x,y)}$. $R := \mathbb{Z}_p[\![x, y]\!]/(p^2 + x^2 + y^2)$ is BCM-regular for $p \gg 0$ by Main Theorem. This fact also follows from [CRMP+21], [MST+22].

A key ingredient in the proof is the Ax-Kochen-Ershov principle, as in [Sch07].

References

[CRMP+21] Javier Carvajal-Rojas, Linquan Ma, Thomas Polstra, Karl Schwede, and Kevin Tucker. Covers of rational double points in mixed characteristic. *J. Singul.*, 23:127–150, 2021. 2

[MS21] Linquan Ma and Karl Schwede. Singularities in mixed characteristic via perfectoid big Cohen-Macaulay algebras. *Duke Math. J.*, 170(13):2815–2890, 2021. 1

[MST+22] Linquan Ma, Karl Schwede, Kevin Tucker, Joe Waldron, and Jakub Witaszek. An analogue of adjoint ideals and PLT singularities in mixed characteristic. *J. Algebraic Geom.*, 31(3):497–559, 2022. 2

[Sch07] Hans Schoutens. Asymptotic homological conjectures in mixed characteristic. *Pacific J. Math.*, 230(2):427–467, 2007. 2